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CONTROLLABILITY AND DECOMPOSITION IN MECHANICAL SYSTEMS†

V. A. VUJIČIĆ and A. M. KOVALEV

Belgrade, Donetsk

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The method of oriented manifolds [1] is used to obtain criteria for the controllability of non-linear mechanical systems of general form. For systems that are linear in control, the question of controllability is reduced to an analysis of the solvability of a system of partial differential equations of special form typical of the invariant relations method and the Lyapunov function method. When the number of controls is equal to the number degrees of freedom, using the example of specific systems with two degrees of freedom the case of confluence of the matrix of coefficients for the control vector in the equations of motion is considered. In the context of a discussion of the property [2] of complete controllability of classes of mechanical systems, problem formulations are proposed in which weakening of the property of control robustness (only variation of the constant parameters is allowed) enables new classes of controllable systems to be obtained. The important case, for mechanics, of the decomposability of the equations of motion into kinematic and dynamic equations is investigated and a theorem establishing the relation between the controllability of the linear system and its dynamical subsystem is proved. Examples are given. The problem of controlling the angular velocity and orientation of a rigid body by means of a single jet engine is considered, for the solution of which the method of oriented manifolds and the decomposition method are used. © 2000 Elsevier Science Ltd. All rights reserved.

1. CONTROLLABILITY OF HOLONOMIC MECHANICAL SYSTEMS

The motion of a controllable, holonomic, scleronomous mechanical system with n degrees of freedom will be described in canonical Hamilton variables p, q:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} + Q_i, \quad i = 1,\ldots,n$$
$$H = \frac{1}{2}a^{ij}(q)p_i p_j + \Pi(q), \quad Q_i = Q_i(q,p,u) \tag{1.1}$$

where $H = H(q, p)$ is the Hamilton function, Q_i are generalized non-potential forces, which depend on the m-dimensional control vector $u \in U$, the canonical variables (q, p) belong to the region $D \subseteq T^*M$ of the cotangential manifold T^*M, and time t varies in the range $T \subseteq (0, \infty)$. The functions II, a^{ij} and Q_i are regarded as functions of their arguments, differentiated a sufficient number of times. In a number of problems, constraining sets U containing controls that are fairly large (in absolute value) controls are considered. In these cases the system will be said to possess sufficiently large control resources. Note that the time derivative of the function $V(q, p)$, specified in solutions of system (1.1), is defined by the formula.

$$\frac{dV}{dt} = [V, H] + \left(\frac{\partial V}{\partial p}, Q\right)$$

We will consider the controllability of system (1.1) in the classical formulation as a property of the system that ensures the existence of an admissible control, under the action of which the system transfers from an arbitrarily specified initial state to an arbitrarily specified final state of motion. For the investigation, we will use a method [1] based on the idea of an oriented manifold (OM).

Definition 1. The manifold $K \subset D$ will be said to be oriented with respect to system (1.1) in region D if it coincides with its positive orbit ($K = \mathrm{Or}^+K$) or its negative orbit ($K = \mathrm{Or}^-K$). The positive orbit Or^+K of the set K is a set of points attainable from set K along trajectories of system (1.1), and the

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negative orbit $\mathrm{Or}^{-}K$ is a set of points from which set K can be reached.

The necessary and sufficient conditions of controllability of system (1.1) are given by the following theorem [1].

Theorem 1. System (1.1) is controllable in region D when, and only when, there are no system-oriented manifolds N with a smooth boundary such that $N \neq \emptyset, D$.

The condition of orientability means that, for OMs of complete dimensionality $k = 2n$, the vector $\partial H/\partial p$ and covector $(-\partial H/\partial q + Q)$ of system (1.1) at points of its boundary are in one direction from it (outward or inward), while for OMs incomplete dimensionality $k < 2n$ the coordinates $\dot{q}, \dot{p}$ of the phase velocity belong to a cotangential manifold at its internal points and are in one direction from it at points of its boundary. Using this property, Theorem 1 enables us to reduce the investigation of the controllability of system (1.1) to a study of the solvability of a system of linear first order partial differential equation.

Theorem 2. Mechanical system (1.1) is controllable in region D when, and only when, the system partial differential equations.

$$
\begin{aligned}
&[V_0, H] + \left(Q, \frac{\partial V_0}{\partial p}\right) = \sum_{j=0}^{2n-1} \lambda_{0j}(q,p,u)V_j + G(q,p,u) \\
&[V_i, H] + \left(Q, \frac{\partial V_i}{\partial p}\right) = \sum_{j=1}^{2n-1} \lambda_{ij}(q,p,u)V_j, \quad i = 1,\dots,2n-1
\end{aligned}
\tag{1.2}
$$

where $G(q, p, u)$ is a constant-sign function in the region $D \times U$, and the function $\lambda_{ij}(q, p, u)$ does not contain singularities in the region $D \times U$, does not have any vanishing solutions $V_j(q, p)$ in region D.

The necessity follows from the fact that, specifying the smooth boundary of OMs by controls $V_i(q, p) = 0 (i = 0, 1, \dots, 2n - 1)$, using the property of orientability, we obtain that the functions V_i satisfy Eqs (1.2).

The proof of sufficiency reduces to the fact that the existence of a solution $V_i(q, p)$ $(i = 0, 1, \dots, 2n - 1)$ leads to the presence of an OM, the boundary of which is defined by the equations $V_i(q, p) = 0$ $(i = 0, 1, \dots, 2n - 1)$, which, by virtue of Theorem 1, leads to non-controllability.

2. SYSTEMS THAT ARE LINEAR IN CONTROL

Theorem 2 reduces the solution of the problem of the controllability of system (1.1) to a study of the existence of a solution of system of differential equations (1.2). The latter problem is complicated by the fact that these equations contain the control parameter u, which can take any values from the set U. When system (1.1) depends linearly on the control

$$
\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} + f_i + \sum_{s=1}^{m} g_{is}u^s, \quad i = 1,\dots,n
\tag{2.1}
$$

instead of Eqs (1.2) it is possible to obtain a system of differential equations containing no control. Here, $f_i(q, p)$ and $g_{is}(q, p)$ are assumed to be functions of their arguments, differentiated a sufficient number of times. Furthermore, we assume that the set U contains a zero as an internal point.

Theorem 3. Mechanical system (2.1) with sufficiently large control resources is controllable in region D when, and only when, the system of partial differential equations

$$
\begin{aligned}
&[V_0, H] + \left(f, \frac{\partial V_0}{\partial p}\right) = \sum_{j=0}^{2n-1} \lambda_{0j}V_j + G_0(q,p) \\
&[V_i, H] + \left(f, \frac{\partial V_i}{\partial p}\right) = \sum_{j=1}^{2n-1} \lambda_{ij}V_j, \quad i = 1,\dots,2n-1 \\
&\left(g_s, \frac{\partial V_l}{\partial p}\right) = \sum_{j=0}^{2n-1} \lambda_{ljs}V_j, \quad l = 0,1,\dots,2n-1, \quad s = 1,\dots,m \\
&f = (f_1,\dots,f_n)^T, \quad g_s = (g_{1s},\dots,g_{ns})^T
\end{aligned}
\tag{2.2}
$$

where $G_0(q, p)$ is a constant-sign function in region D, while the functions λ_{ij} and λ_{ljs} contain no singularities in region D, and $\lambda_{l0s} = 0$ for $l > 0$, has no vanishing solutions $V_j(q, p)$ in region D.

Proof. The necessity will be proved by reasoning from contradiction. In spite of the assertion of the theorem, let system (2.2) have the solution $V_i = V_i(q, p)$ $(i = 0, \ldots, 2n-1)$. We multiply the last group of Eqs (2.2) by u^s and sum over s from 1 to m. We add sum obtained for $l = 0$ to the first equation, and for $l = 1, \ldots, 2n-1$ to the first group of Eqs (2.2). As a result we obtain that the functions V_i determine the solutions of system (1.2), in which

$$Q_i = f_i + \sum_{s=1}^{m} g_{is}u^s, \quad \lambda_{ij}(q,p,u) = \lambda_{ij} + \sum_{s=1}^{m} \lambda_{ijs}u^s, \quad G(q,p,u) = G_0(q,p)$$

On the basis of Theorem 2, we conclude that system (2.1) is not controllable, which contradicts the condition and proves the necessity.

We shall also prove the sufficiency by reasoning from contradiction. In spite of the assertion of the theorem, let system (2.2) be uncontrollable in region D. Then, on the basis of Theorem 2, system (1.2) for

$$Q_i = f_i + \sum_{s=1}^{m} g_{is}u^s$$

has the solution $V_j = V_j(q, p)$ $(j = 0, 1, \ldots, \alpha, \alpha \leq 2n-1)$, and here V_j and also λ_{ij} and G are functions of their arguments, differentiated a sufficient number of times. Assuming in Eqs (1.2) that $u^s = 0$, we establish that the given functions V_j satisfy the first equation and the first group of Eqs (2.2) in which $\lambda_{ij} = \lambda_{ij}(q, p, 0)$ and $G_0(q, p) = G(q, p, 0)$. Since the region U can be selected to be fairly large, a point $u_0 \in U$ exists in which the constant-sign function $G(q, p, u)$ takes an extremum. In the vicinity of the point u_0, the expansions

$$\begin{gathered} G(q,p,u) = G(q,p,u_0) + \sum_{ij=1}^{m} G_{ij}(q,p,u_0)(u^i - u_0^i)(u^j - u_0^j + \ldots \lambda_{ij}(q,p,u)) = \lambda_{ij}(q,p,u_0) + \\ + \sum_{s=1}^{m} \lambda_{ijs}(q,p,u_0)(u^s - u_0^s) + \ldots \end{gathered} \tag{2.3}$$

hold, where the unwritten terms are higher-order infinitesimals.

Substituting expressions (2.3) into Eqs (1.2), we obtain

$$\begin{gathered} [V_0, H] + \left(f + \sum_{s=1}^{m} g_s u_0^s, \frac{\partial V_0}{\partial p} \right) - \sum_{j=0}^{\alpha} \lambda_{0j}(q,p,u_0)V_j - G(q,p,u_0) + \\ + \left(\sum_{s=1}^{m} g_s(u^s - u_0^s), \frac{\partial V_0}{\partial p} \right) - \sum_{j=0}^{\alpha} \sum_{s=1}^{m} \lambda_{0js}(q,p,u_0)(u^s - u_0^s)V_j + \ldots = 0 \\ [V_i, H] + \left(f + \sum_{s=1}^{m} g_s u_0^s, \frac{\partial V_i}{\partial p} \right) - \sum_{j=1}^{\alpha} \lambda_{ij}(q,p,u_0)V_j + \left(\sum_{s=1}^{m} g_s(u^s - u_0^s), \frac{\partial V_i}{\partial p} \right) - \\ - \sum_{j=1}^{\alpha} \sum_{s=1}^{m} \lambda_{ijs}(q,p,u_0)(u^s - u_0^s)V_j + \ldots = 0 \end{gathered} \tag{2.4}$$

Since Eqs (1.2) are satisfied at the point $u = u_0$ and the variables u^s are arbitrary, it follows from Eqs (2.3) that

$$\left(g_s, \frac{\partial V_l}{\partial p} \right) = \sum_{j=0}^{\alpha} \lambda_{ljs}(q,p,u_0)V_j, \quad l = 0, 1, \ldots, \alpha$$

where $\lambda_{l0s} = 0$ for $l > 0$, i.e. the functions V_j also satisfy the last group of Eqs (2.2). Hence, we have established that system (2.2) allows if the solution $V_j = V_j(q, p)$, which contradicts the condition and proves the theorem.

Remark 1. The condition of Theorem 3 concerning the possibility of the control modulus taking sufficiently large values is extremely important, and its non-satisfaction can lead to uncontrollability of the system. Thus, for the system $\dot{x}_i = \lambda_i x_i + u$ ($x \in R^n$, $u \in R^1$) with the condition $\lambda_i \neq \lambda_j$, the Kalman criterion is satisfied, and the given system is controllable. However, with the constraints

$$\lambda_i > 1, x \in D = \{x : \|x\| < R, R > 2\}, \quad u \in U = \{u : -1 < u < 1\}$$

the system is uncontrollable in region D (with $u \in U$) in view of the obvious estimate $\dot{x}_i > 0$ for $x_i > 1$ and any $u \in U$. On the other hand, this condition is not necessary; a Kalman controllable system $\dot{x} = Ax + Bu$, when the eigenvalues of matrix A are pure imaginary, is controllable in any sphere $\|x\| < R$ for controls $\|u\| < \varepsilon$ [3] as small as desired.

3. THE CASE OF COMPLETE CONTROL ($m = n$)

For the case of complete control, when the dimensionality of the control matches the dimensionality of the coordinate space $m = n$, by using Theorem 3 and the kinematic properties of holonomic systems it is possible to obtain simple sufficient conditions of controllability. To illustrate this we present the following theorem which stems from the results of [2].

Theorem 4. Let $m = n$ and $\det\|g_{is}\| \neq 0$ in region D. Then, with sufficiently large control resources, system (2.1) is controllable in region D.

Proof. On the basis of Theorem 3 it is sufficient to show that, when the conditions of Theorem 4 are satisfied, system (2.2) has no vanishing solutions. If such a solution exists, then from the last group of Eqs (2.2) it follows [4, 5] that systems of equations $\dot{q} = 0, \dot{p} = g_s(q, p)$ ($s = 1, \ldots, n$) have a common invariant manifold M defined by the equations $V_j = 0$ ($j = 0, 1, \ldots, \alpha$, $\alpha \leqslant 2n - 1$). Since at each point of the manifold M the normal to it is orthogonal to the velocity vectors $[0, g_s(q, p)]$ and the condition $\det\|g_{is}\| \neq 0$ is satisfied, we obtain $\partial V_j/\partial p = 0$, i.e. $V_j = V_j(q)$. The remaining equations of system (2.2) take the form

$$[V_0, H] = \sum_{j=0}^{\alpha} \lambda_{0j} V_j + G_0(q, p), \quad [V_i, H] = \sum_{j=1}^{\alpha} \lambda_{ij} V_j, \quad i = 1, \ldots, \alpha \tag{3.1}$$

From Eqs (3.1) it follows that the manifold M is the OM for the Hamiltonian system with the function $H(q, p)$. By virtue of the property $V_j = V_j(q)$, the manifold M is a cylinder, the generator of which is identical with momentum space. Therefore, at points $(g_0, p) \in M$ for any p from the region of values, the normal to the manifold M is the same. From the definition of holonomic mechanical systems it follows that, at each point q of configuration space, the velocity vectors $\dot{q}$ can take any direction. By virtue of the mutual uniqueness of the mapping $\dot{q} \to p$, it follows from this that, when p varies in any neighbourhood as small as desired, the corresponding velocities $\dot{q}$ can take any direction. Therefore, for the normal n to manifold M at any point (q_0, p_0) there are points $(q_0, p) \in M$ with the same normal n at which the projection of the velocity $\dot{q} = \partial H/\partial p$ onto the normal n takes both positive and negative values. This means that the manifold M cannot be oriented, and Eqs (3.1), and consequently Eqs (2.2), have no vanishing solutions in region D.

Remark 2. This theorem can be proved by different methods. Thus, in [2] it is proved using discontinuous controls of special form on the basis of the theory of differential equations with a discontinuous right-hand side. Provided $m = n$, system (2.1) is a system of "triangular form", and it can be shown that, when the conditions of the theorem are satisfied, sufficient conditions of controllability [4, 6] are satisfied for it. The proof of this theorem by the inverse-problem method is of interest. Taking account of the uniqueness of the mapping $\dot{q} = a^T p$, it is sufficient to obtain an expression for the control vector in terms of the coordinate vector and its derivatives. Differentiating the first group of Eqs (2.1) with respect to time, we find

$$\ddot{q}_i - \sum_{j=1}^{n} \left[\frac{\partial^2 H}{\partial p_i \partial p_j} \left(-\frac{\partial H}{\partial q_j} + f_j \right) + \frac{\partial^2 H}{\partial p_i \partial q_j} \dot{q}_j \right] = \sum_{s,j=1}^{n} \frac{\partial^2 H}{\partial p_i \partial p_j} g_{js} u^s, \quad i = 1, \ldots, n \tag{3.2}$$

Here, in place of the momenta p_i, we have substituted their expressions in terms of q, $\dot{q}$, as given by the formulae $p_i = \partial L/\partial \dot{q}_i$. By virtue of the fact that

$$\det\left\|\frac{\partial^2 H}{\partial p_i \partial p_j}\right\| = \|a^{ij}\| \neq 0$$

and by the condition $\det\|g_{is}\| \neq 0$, the system of equations (3.2) is uniquely solvable for the controls $u_s = u_s(q, \dot{q}, \ddot{q})$. For the specified boundary values (q_0, p_0), (q_1, p_1), from the first group of Eqs (2.1) we find $\dot{q}_0$, $\dot{q}_1$. Selecting an arbitrary function $\tilde{q}(t)$, twice continuously differentiable, and satisfying the conditions $\tilde{q}(t_0) = q_0$, $\tilde{q}(t_1) = q_1$, $\dot{\tilde{q}}(t_0) = \dot{q}_0$, $\dot{\tilde{q}}(t_1) = \dot{q}_1$, we determine the functions $\tilde{u}_s(t) = u_s(\tilde{q}(t), \dot{\tilde{q}}(t), \ddot{\tilde{q}}(t))$ which provide a solution to the boundary value problem in question. Note that, besides the proof of controllability, this also provides a proof of the functional controllability with respect $q \in C^2$, i.e. the possibility of realizing for system (2.1), any function $q(t)$ twice continuously differentiable (by selecting the appropriate control $u(t)$).

The condition $\det\|g_{ij}\| \neq 0$ is not necessary. When it is violated the system may be both controllable and uncontrollable, which is indicated by the following two examples, in the solution of which dimensionless variables are used.

Example 1. Consider the controllability of the system

$$\dot{q}_1 = p_1, \quad \dot{p}_1 = q_1 p_2 + q_1 u_1, \quad \dot{q}_2 = p_2, \quad \dot{p}_2 = q_2 p_1 + u_2 \tag{3.3}$$

with the constraints

$$q_1^2 + q_2^2 + p_1^2 + p_2^2 < r^2, \quad u_1^2 + u_2^2 < r_u^2 \tag{3.4}$$

The sufficient condition of Theorem 4 is not satisfied, since the determinant $\det\|g_{ij}\| = q_1$ can vanish in region D. However, it can be verified that system (3.3) has an invariant manifold defined by the relations $q_1 = 0$ and $p_1 = 0$, i.e. the functions $V_0 = q_1$ and $V_1 = p_1$ are solutions of system (2.2) with $G_0 = 0$. By Theorems 2 and 3, this indicates the uncontrollability of system (3.3) in the region examined.

Example 2. Consider the controllability of the system

$$\dot{q}_1 = p_1, \quad \dot{p}_1 = q_1 p_2 + u_1, \quad \dot{q}_2 = p_2, \quad \dot{p}_2 = q_2 p_1 + q_1 u_2 \tag{3.5}$$

with constraints (3.4)

We write Eqs (2.2) for system (3.5), assuming that $g_1 = (0, 0, 1, 0)$ and $g_2 = (0, 0, 0, q_1)$, we obtain

$$\begin{gathered} FV_i = \sum_{j=0}^{3} \lambda_{0j} V_j + G_0, \quad FV_i = \sum_{j=1}^{3} \lambda_{ij} V_j, \quad i = 1,2,3 \\ \left(F = p_1 \frac{\partial V_i}{\partial q_1} + p_2 \frac{\partial V_i}{\partial q_2} + q_1 p_2 \frac{\partial V_i}{\partial p_1} + q_2 p_1 \frac{\partial V_i}{\partial p_2} \right) \\ \frac{\partial V_i}{\partial p_1} = \sum_{j=0}^{3} \lambda_{ij1} V_j, \quad q_1 \frac{\partial V_i}{\partial p_2} = \sum_{j=0}^{3} \lambda_{ij2} V_j, \quad i = 0,1,2,3 \end{gathered} \tag{3.6}$$

As in the previous example, we have $\det\|q_{ij}\| = q_1$. If $\det\|q_{ij}\| \neq 0$, system (3.6) has no vanishing solutions. It remains to be verified whether system (3.6) allows of a solution V_0, V_1, V_2, V_3 containing the function $V_1 = q_1$. Substituting function V_1 into the second group of Eqs (3.6), we find

$$p_1 = \sum_{j=1}^{3} \lambda_{1j} V_j$$

It follows that there is one further function $V_2 = p_1$ and $\lambda_{12} = 1$, and the remaining ones $\lambda_{1j} = 0$. Substituting the function V_2 into the third group of Eqs (3.6), we obtain

$$1 = \sum_{j=0}^{3} \lambda_{2j1} V_j$$

It follows that system (3.6) has no vanishing solutions, and system (3.5) is controllable.

4. THE CONTROLLABILITY OF CLASSES OF MECHANICAL SYSTEMS

To simplify the solution of the problem of the controllability of a specific mechanical system under conditions of incomplete information concerning the active forces and parameters, the concept of the controllability of classes of mechanical systems was introduced in [2]. The condition of controllability of a class guarantees retention of this property when there are variations of the parameters and forces, i.e. it has a robust nature, which makes it possible to reveal general laws that are free from the individual singularities of the specific system. The class of systems in [2] is described by the sets U, D_0 of values of the controls and generalized forces and by numbers b_0, λ_0, λ_1 characterizing the kinetic energy and a matrix for the control vector in the equations of motion. The simplest class was singled out, described by the equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i}-\frac{\partial T}{\partial q_i}=u_i(t),\quad |u_i|\leqslant h_i,\quad i=1,\ldots,n \tag{4.1}$$

for which it was established that it is completely controllable when, and only when,

$$h_0=\min_{1\leqslant i\leqslant n} h_i>0$$

Analysing the properties of systems of this class, we shall point out two of the most remarkable: (1) systems of this class are controllable by arbitrarily small controls; (2) the property of controllability does not depend on the properties of the system (on the expression for the kinetic energy), i.e. the constraints on the class are minimal.

Further, extensions of this class, where considered in [2] when the right-hand sides of Eqs (4.1) have the form

$$\text{(a)}\ Q_i(q,\dot{q},t)+u_i$$

$$\text{(b)}\ Q_i(q,\dot{q},t)+\sum_{k=1}^{n} b_{ik}(q,\dot{q},t)u_k(t)$$

The conditions of controllability of these classes include a condition ensuring the choice of a control that not only compensates for the generalized forces but also guarantees a resource ensuring complete controllability, and for classes b, the further condition

$$\det\|b_{ij}(q,\dot{q},t)\|\neq 0,\quad (q,\dot{q})\in R^{2n},\quad t\geqslant t_0 \tag{4.2}$$

Comparing these classes with the simplest, we note that they no longer possess the property of controllability by small controls, and for class b the additional condition (4.2) arises. Loss of the first property is extremely important from the viewpoint of applications, and at the same time condition (4.2) is not so constraining if the problem of describing the classes is examined in certain spaces, in the simplest case in the space of parameters which specify the kinetic energy and generalized forces. Then the simplest class is described by the manifold $Q(q, \dot{q}, t) = 0$, and condition (4.2) isolates a certain region (full measure) in which the parameters can vary arbitrarily.

An examination of classes a and b together with the simplest class leads to the formulation of the problem of finding the classes of controllable mechanical systems for which retention of the property of controllability is guaranteed when of the parameters vary on certain manifolds (for the simplest class this is the manifold $Q = 0$, and for class b it is the region isolated by condition (4.2). The solution of this problem can be obtained using the theory of the control of systems with control constraints. Here, the conditions of controllability provide a description of the classes (in the corresponding spaces). Thus, for linear mechanical systems and more common linear dynamical systems, a description of the class of systems that are controllable by controls as small as desired is given by the following theorem [3].

Theorem 5. The system

$$\dot{x}=Ax+bu,\quad x\in R^n,\quad u\in R^1,\quad |u|<u_0$$

is controllable when, and only when, $\det(b, Ab, \ldots, A^{n-1}b)\neq 0$ and all the eigenvalues of the matrix A are pure imaginary (including zero eigenvalues), and here the value of u_0 can be taken to be as small as desired.

For systems in which the control is linear, this problem can be solved using Theorem 3, and for non-linear systems of general form it can be solved using Theorem 2. Note that, in this formulation of the problem of the controllability of classes of mechanical systems, the condition that the dimensionality of the control should be identical with the dimensionality of the coordinate space ($m = n$) ceases to be a necessary condition of controllability, and the properties of controllability "proper" acquire a decisive value, making it possible to act in a desired way on the multidimensional system by a control of smaller dimensionality, even a one-dimensional control, as in Theorem 5.

5. THE CONTROLLABILITY OF DECOMPOSABLE SYSTEMS

The non-linearity, large dimensionality and relation between different degrees of freedom that are characteristic of the mechanical systems describing modern technical plant make it considerably more difficult to study their dynamic properties, including their controllability. In developing methods of controllability, ideas of decomposition have proved to be extremely effective. These basically reduce the investigation of the high-order system to an investigation of several lower-order systems by dividing the initial system into independent subsystems or by introducing a certain hierarchical structure.

One of the implementations of the first approach for mechanical systems is the principle of decomposition [7]. In the theory of the control of dynamical systems, the problem of decomposition has been well studied in a general formulation, a fairly complete presentation of which is given in [8]. Analysing controlled mechanical systems from these viewpoints, it is possible to note their initial state of decomposition in relation to the controls. The equations of motion of mechanical systems can be naturally divided into kinematic or kinetic (not containing a control) and dynamic equations, depending on the controls. For canonical variables, these are respectively the first and second group of equations (1.1). The equations in Lagrangian variables have an even simpler structure, namely,

$$\dot{q}^i = v^i, \quad \dot{v}^i = F^i(t, q, v, u), \quad i = 1, \ldots, n \tag{5.1}$$

In order to simplify the investigation, other groups of variables are often introduced in mechanics, retaining the division of the equations into dynamic and kinematic equations. A classical example is provided by the Euler–Poisson equations of motion of a rigid body with fixed point in a gravitational field, where the projections of the angular velocity vector of the body onto moving axes are adopted as the dynamic variables, and either the direction cosines of the vertical or the Euler angles can be adopted as kinematic variables.

A more extensive decomposition arises when the dynamic equations do not depend on the kinematic variables, and a hierarchical structure emerges. A study of the controllability of the entire system begins with the solution of the simpler problem of the controllability of its dynamical subsystem, which is often of important independent value. The question of the relation between the properties of controllability of the entire system and its dynamical subsystem is of theoretical and practical interest. The complete solution of this problem can be obtained for linear scleronomous systems of the type (5.1) using the following theorem.

Theorem 6. The mechanical system

$$\dot{q} = v, \quad \dot{v} = Av + Bu \quad (q, v) \in R^{2n}, \; u \in R^m \tag{5.2}$$

is controllable when, and only when, $m = n$ and $\det B \neq 0$.

Proof. The necessity will be proved by reasoning from contradiction. In spite of the assertion, either $m < n$ or $\det B = 0$. A non-zero vector c then exists orthogonal to the vectors $b_1, \ldots, b_m$, where $B = (b_1, \ldots, b_m)$. For the linear function $V = (v - Aq)^T c$, the equality $\dot{V} = 0$ is satisfied. This means that the function V is an integral of system (5.2) and, consequently, system (5.2) is uncontrollable, which contradicts the condition and proves the necessity.

Sufficiency follows from the fact that, when the conditions of the theorem are satisfied, system (5.2) has a "triangular form" and the sufficient conditions of controllability [4, 6] are satisfied for it. Use can also be made of Kalman's criterion, which for system (5.2) leads to the condition.

$$\det\begin{Vmatrix} 0 & B \\ B & AB \end{Vmatrix} = (-1)^n (\det B)^2 \neq 0$$

When the conditions of Theorem 6 are satisfied, its dynamical subsystem is controllable. However, the controllability of the dynamical subsystem alone is insufficient for the controllability of the entire system: the conditions of Theorem 6 must be satisfied. The requirement of complete control $m = n$ is very severe and not obligatory for linear subsystems with general kinematic equations arising in the theory of perturbations of mechanical systems when studying vibrations, problems of stability, etc., i.e. for subsystems of the form.

$$\dot{q} = Cq + Dp, \quad \dot{p} = Ap + Bu, \quad (q, p) \in R^{2n}, \quad u \in R^m \tag{5.3}$$

As an example, we will consider two mechanical systems with two degrees of freedom, using dimensionless variables.

Example 3. Consider a mechanical system with two degrees of freedom with generalized coordinates q_1, q_2, generalized velocities υ_1, υ_2, kinetic energy $T = (\upsilon_1^2 + \upsilon_2^2)/2$ and generalized forces $Q_1 = \upsilon_2$, $Q_2 = u$, where u is the control. We write the equations of motion in the form

$$\dot{q}_1 = \upsilon_1, \quad \dot{q}_2 = \upsilon_2, \quad \dot{\upsilon}_1 = \upsilon_2, \quad \dot{\upsilon}_2 = u \tag{5.4}$$

The dynamical subsystem is controllable. The entire system is uncontrollable since $m = 1 < 2 = n$. The reason for uncontrollability is the presence of the integral $V = \upsilon_1 - q_2 =$ const. Note that the rank of the matrix of controllability of the entire system is equal to 3, and on integral manifolds the entire system is controllable.

Example 4. Consider a mechanical system with two degrees of freedom with generalized coordinates q_1, q_2, kinetic energy $T = (\dot{q}_1^2 + \dot{q}_2^2)/2 + 2q_2\dot{q}_2$ and generalized forces $Q_1 = \dot{q}_2 + q_2$, $Q_2 = -q_2 + u$, where u is the control. In Lagrangian variables, the dynamical subsystem depends on the generalized coordinate q_2 and does not have the form (5.2). In canonical variables

$$q_1, q_2, p_1 = \partial L / \partial \dot{q}_1 = \dot{q}_1, p_2 = \partial L / \partial \dot{q}_2 = \dot{q}_2 + q_2$$

the equations of motion

$$\dot{q}_1 = p_1, \quad \dot{q}_2 = p_2 - q_2, \quad \dot{p}_1 = p_2, \quad \dot{p}_2 = u \tag{5.5}$$

have the form (5.3), differing from Eqs (5.2) in the structure of the kinematic equations. Applying Kalman's criterion to system (5.5), we can verify that it is controllable. The dynamical subsystems of systems (5.4) and (5.5) are identical and are controllable, and, unlike uncontrollable system (5.4), a change in the kinematic equations has led to the controllability of the entire system (5.5).

The examples considered indicate the importance of formulating the problem of the relationship between the properties of the controllability of the mechanical system and of its dynamical subsystem. For linear systems in Lagrangian variables, this problem is solved entirely on the basis of Theorem 6. To investigate linear systems in arbitrary variables, it is necessary to investigate Eqs (5.3). It is of interest to formulate this problem for systems with linear control and for non-linear systems of general form. An interesting example of a non-linear mechanical system that has linear control and has a decomposed dynamical subsystem is provided by the problem of control of the orientation of a rigid body by means of a reaction force, examined in the following section.

6. CONTROL OF THE ORIENTATION OF A RIGID BODY

Many aspects of the motion of a rigid body about its centre of mass under the action of a reaction force are investigated using a model of an absolutely rigid body ignoring the change in mass. The equations of motion relative to a certain inertial system of coordinates have the form

$$\begin{gathered}
\dot{\varphi} = -(\omega_1 \sin\varphi + \omega_2 \cos\varphi)\operatorname{ctg}\theta + \omega_3 \\
\dot{\psi} = (\omega_1 \sin\varphi + \omega_2 \cos\varphi)/\sin\theta, \quad \dot{\theta} = \omega_1 \cos\varphi - \omega_2 \sin\varphi \\
\dot{\omega}_1 = a_1\omega_2\omega_3 + \alpha_1 u \ (123) \ \left(a_1 = (A_2 - A_3)/A_1 \ (123)\right)
\end{gathered} \tag{6.1}$$

where φ, ψ and θ are the Euler angles, ω_1, ω_2 and ω_3 are the projections of the angular velocity vector onto the principal central axes, A_1, A_2 and A_3 are the principal central moments of inertia of the body,

$\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is the unit vector of the direction of the moment of the reaction force, u is the control, characterizing the magnitude of the moment of the reaction force, and the symbol (123) denotes that the remaining unwritten relations are obtained from the written relation by cyclic permutation of the subscripts.

Equations (6.1) can be split into kinematic equations for the Euler angles and dynamic equations with the moments of the forces. Therefore, it is natural to begin our investigation of system (6.1) with an analysis of the dynamic equations, and then investigate the entire system and analyse the question of the relation between the properties of controllability of the entire system and of its dynamical subsystem. The controllability of the dynamic subsystem was investigated in [1, 4, 9], where its controllability was established provided the parameters did not satisfy the conditions of one of the following groups

$$\alpha_1 = \alpha_2 = 0; \ \alpha_1 = 0, \ a_1 = 0; \ a_1\alpha_3^2 - a_3\alpha_1^2 = 0 \ (123) \tag{6.2}$$

We shall obtain the necessary conditions of controllability of the entire system (6.1), using Theorem 3 for the single function $V_0 = V$ and $G_0 = 0$. Equations (2.2) for system (6.1) have the form

$$\begin{aligned} G_1 &= \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 + [\omega_3 - (\omega_1 \sin\varphi + \omega_2 \cos\varphi)\operatorname{ctg}\theta]q_1 + \\ &+ \frac{\omega_1 \sin\varphi + \omega_2 \cos\varphi}{\sin\theta} q_2 + (\omega_1 \cos\varphi - \omega_2 \sin\varphi)q_3 - \lambda_1 V = 0 \\ G_2 &= \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 - \lambda_2 V = 0 \\ &\left(p_i = \frac{\partial V}{\partial \omega_i}, \ i = 1,2,3; \ q_1 = \frac{\partial V}{\partial \varphi}, \ q_2 = \frac{\partial V}{\partial \psi}, \ q_3 = \frac{\partial V}{\partial \theta}; \ \beta_1 = a_1\omega_2\omega_3 \quad (123) \right) \end{aligned} \tag{6.3}$$

To obtain the conditions of solvability of this system, we shall supplement it with Jacobi brackets. We arrive at the following system

$$G_i = 0, \ i = 1, \ldots, 6 \tag{6.4}$$

Here

$$\begin{aligned} G_3 &= [G_2, G_1] = \xi_1 p_1 + \xi_2 p_2 + \xi_3 p_3 + [\alpha_3 - (\alpha_1 \sin\varphi + \alpha_2 \cos\varphi)\operatorname{ctg}\theta]q_1 + \\ &+ \frac{\alpha_1 \sin\varphi + \alpha_2 \cos\varphi}{\sin\theta} q_2 + (\alpha_1 \cos\varphi - \alpha_2 \sin\varphi)q_3 - \lambda_3 V \\ G_4 &= [G_2, G_3] = \eta_1 p_1 + \eta_2 p_2 + \eta_3 p_3 - \lambda_4 V \\ G_5 &= [G_4, G_1] = \zeta_1 p_1 + \zeta_2 p_2 + \zeta_3 p_3 + \\ &+ 2[a_3\alpha_1\alpha_2 - \alpha_3(a_1\alpha_2 \sin\varphi + a_2\alpha_1 \cos\varphi)\operatorname{ctg}\theta]q_1 + \\ &+ 2\alpha_3 \frac{a_1\alpha_2 \sin\varphi + a_2\alpha_1 \cos\varphi}{\sin\theta} q_2 + 2\alpha_3(a_1\alpha_2 \cos\varphi - a_2\alpha_1 \sin\varphi)q_3 - \lambda_5 V \\ G_6 &= [G_4, G_3] = \varkappa_1 p_1 + \varkappa_2 p_2 + \varkappa_3 p_3 - \lambda_6 V \\ \xi_1 &= a_1(\alpha_2\omega_3 + \alpha_3\omega_2), \quad \eta_1 = 2a_1\alpha_2\alpha_3 \quad (123) \\ \zeta_1 &= 2a_1\alpha_1(a_2\alpha_3\omega_3 + a_3\alpha_2\omega_2), \quad \varkappa_1 = 2a_1\alpha_1(a_2\alpha_3^2 + a_3\alpha_2^2) \quad (123) \end{aligned}$$

and $\lambda_1, \ldots, \lambda_6$ are certain functions of the variable ω_1, ω_2, ω_3, φ, ψ, and θ. The determinant of system (6.4), regarded as a system of linear algebraic equations in p_i, q_i ($i = 1, 2, 3$), is equal to

$$\begin{aligned} &\Delta = 8\,\delta_1\delta_2\delta_3 W/\sin\theta, \quad W = \alpha_1\delta_1\omega_1 + \alpha_2\delta_2\omega_2 + \alpha_3\delta_3\omega_3 \\ &\delta_1 = a_2\alpha_3^2 - a_3\alpha_2^2 \quad (123) \end{aligned} \tag{6.5}$$

If $\Delta \neq 0$, then, as shown in [4], system (6.4), together with system (6.3), either has no solution or has a solution of exponential form which dose not vanish. It remains to study the case $\Delta = 0$. Determinant

(6.5) is zero when the parameters a_i, α_i (1 = 1, 2, 3) satisfy conditions (6.2), and on a linear manifold

$$W = 0 \tag{6.6}$$

When conditions (6.2) are satisfied, system (6.3) obviously has a solution since in this case the dynamical subsystem of system (6.1) is uncontrollable. To satisfy (6.6) it is necessary for the function W to be a solution of system (6.3). By substituting it into (6.3), we can verify that the second equation is satisfied identically of $\lambda_2 = 0$, while the first equation acquires the form

$$a_1\omega_2\omega_3 \frac{\partial W}{\partial \omega_1} + a_2\omega_3\omega_1 \frac{\partial W}{\partial \omega_2} + a_3\omega_1\omega_2 \frac{\partial W}{\partial \omega_3} = \lambda_1 W \tag{6.7}$$

Using the Levi-Civita theorem [5], Eqs (6.7) indicates that linear manifold (6.6) is an invariant manifold of the system of equations

$$\dot{\omega}_1 = a_1\omega_2\omega_3 \ (123) \tag{6.8}$$

which describes the motion of a rigid body by inertia. From the dynamics of a rigid body it is well known that Eqs (6.8) admit of linear invariant manifolds only of the following form

$$\omega_1\sqrt{a_2} \pm \omega_2\sqrt{a_1} = 0 \ (123)$$

The requirement that Eqs (6.9) and (6.6) should be identical leads to conditions imposed on the parameters

$$a_1\alpha_2^2 - a_2\alpha_1^2 = 0 \ (123)$$

and have already been singled out by equalities (6.2).

Thus, the necessary conditions of controllability of the entire system (6.1) are identical with the necessary (and sufficient) conditions of controllability of its dynamical subsystem, i.e. they are satisfied for all values of the parameters a_i and α_i in addition to those satisfying conditions (6.2).

REFERENCES

1. KOVALEV, A. M., A criterion of controllability and the sufficient conditions of stabilizability of dynamical systems. *Prikl. Mat. Mekh.*, 1995, **59**, 3, 401–409.
2. PYATNITSKII, Ye. S., A criterion of complete controllability of classes of mechanical systems with constrained controls. *Prikl. Mat. Mekh.*, 1996, **60**, 5, 707–718.
3. KOVALEV, A. M., Oriented manifolds and the controllability of dynamical systems. *Prikl. Mat Mekh.*, 1991, **55**, 4, 639–646.
4. KOVALEV, A. M., *Non-linear Problems of Control and Observation in the Theory of Dynamical Systems*. Naukova Dumka, Kiev. 1980.
5. LEVI-CIVITA, T. and AMALDI, U., *Lezioni di Meccanica Razionale*. Zanichelli, Bologna, 1927.
6. KOROBOV, V. I., The controllability and stability of certain non-linear systems. *Differents. Uraveniya*, 1973, **9**, 4, 614–619.
7. PYATNITSKII, Ye. S., The principle of decomposition in the control of mechanical systems. *Dokl. Akad. Nauk SSSR*, 1988, **300**, 2, 300–303.
8. PAVLOVSKII, Yu. N., The theory of factorization and decomposition of controlled dynamical systems and its applications. *Izv. Akad. Nauk SSSR. Tekhn. Kibernetika*, 1984, **2**, 45–57.
9. AGRACHEV, A. A. and SARYCHEV, A. V., Control of the rotation of an asymmetrical rigid body. *Problemy Upravleniya i Teorii Informatsii*, 1983, **12**, 5, 335–347.

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